Memorandum

To: AMCS Group
From: Pedro Elosegui and Per Jarlemark
Subject: Geometry of the “Andersen Manufacturing” AMCS antenna system
Date: July 10, 2000

Geometric model of the AMCS antenna

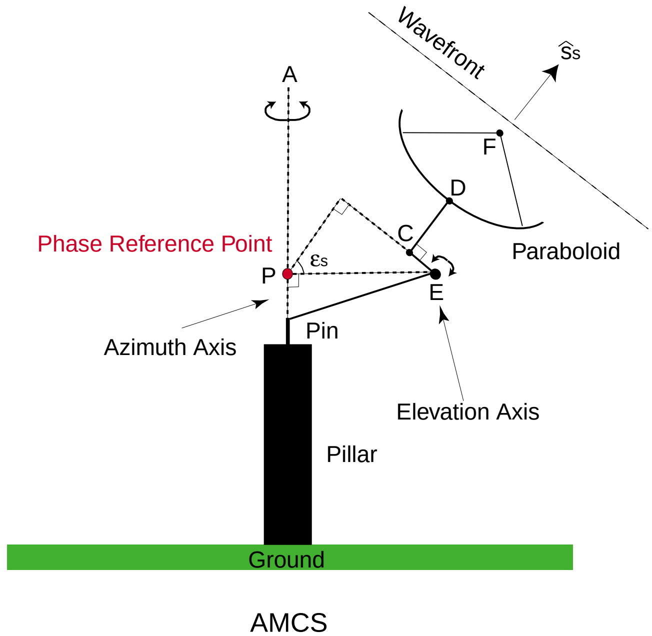

Figure 1 Schematic representation of the (not to scale) geometry of the AMCS antenna model.

Figure 1 shows the geometric model of the “Andersen Manufacturing” AMCS antenna, ie., the “Andersen 3.0-meter true focus paraboloid antenna with motorized dual-drive azimuth-elevation mount and 360-degree azimuth travel” manufactured by Andersen Manufacturing in Idaho Falls, Idaho, that has been operational at Haystack since the Spring of 1999 (hereafter referred to as AMCS antenna). $\hat{s}_s$ is the unit vector in the hypothetical direction (elevation ε_s) of the GPS satellite. The wavefront, an isophase plane, is perpendicular to $\hat{s}_s$. F is the point fixed to the antenna that the AMCS phase differs by a constant phase only. The segment CD is parallel to $\hat{s}_s$ and represents the axis of revolution of the paraboloid. This axis does not intersect the elevation axis, which is point E. The axis of revolution is offset from the elevation axis by a distance CE, the “elevation axis offset”. Point E is the end view of the elevation axis. Thus, the elevation axis is perpendicular to the page and allows rotations on the plane of the page. The elevation axis is offset from the azimuth axis by a horizontal distance EP, the "azimuth axis offset". The azimuth axis contains the segment PA, is on the plane of the page and allows rotations on a plane perpendicular to the page. All the points on the azimuth axis are fixed relative to the ground. A plane that contains the elevation axis and is perpendincular to the azimuth axis intersects the latter at P. For convenience, we define P as the phase reference point of the AMCS antenna.

Geometric model of the AMCS baseline

Figure 2 shows the geometric model of the AMCS baseline. Red dots represent the phase reference points of the GPS and the AMCS antennae. (The position of the GPS antenna phase reference point is typically a few mm different between the L1 and L2 wavelengths, and it is very dependent on the antenna-type used. The figure shows only one GPS phase reference point for simplicity.) The baseline vector between the antennae phase-reference points is $\bar{b}$. The vector $\bar{a}$ is the "true" vector reflected in the AMCS phase difference that we must correct to the phase-reference point. The vectors $\bar{a}$ and $\bar{b}$ are related by

$$\bar{a} = \bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f} \tag{1}$$

where $\bar{c}$ and $\bar{d}$ are the azimuth and elevation axis offsets, respectively.

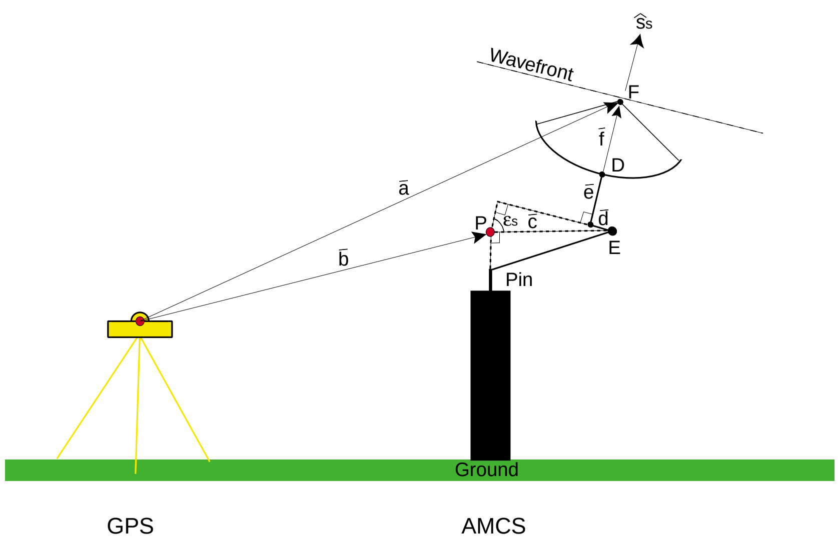

Figure 2 Schematic representation of the geometry of the AMCS baseline model; perfect pointing.

Perfect AMCS pointing

Perfect AMCS pointing occurs when the AMCS antenna pointing direction and the direction of the satellite are identical. Figure 2 illustrates this case. Vector $\hat{s}_s$ is parallel to $\bar{e}$ and $\bar{f}$ and perpendicular to $\bar{d}$. The AMCS phase delay is (distance units):

$$\hat{s}_s \cdot \bar{a} = \hat{s}_s \cdot (\bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f}) = \hat{s}_s \cdot \bar{b} + |\bar{c}| \cos \varepsilon_s + 0 + |\bar{e}| + |\bar{f}| \tag{2}$$

The term $|\bar{e}| + |\bar{f}|$ is constant if we can assume that the instrumental delay of the AMCS antenna system is independent of the antenna pointing position. Thus, this delay is indistinguishable from a clock offset and is absorbed by the AMCS model term that accounts for a constant phase offset between receivers.

"Deliberate" AMCS mispointing

For convenience, we can decompose any mispointing in its azimuth $\Delta\phi$ and elevation $\Delta\varepsilon$ mispointing angles. Figure 3 shows the geometric model of the AMCS antenna for a mispointing in elevation angle. The unit vectors $\hat{s}_a$ and $\hat{s}_s$ are the pointing directions of the AMCS antenna and the GPS satellite, respectively. The angle $\Delta\varepsilon = \varepsilon_a - \varepsilon_s$ between the two unit vectors is the elevation mispointing angle. Similarly, an azimuth mispointing angle $\Delta\phi = \phi_a - \phi_s$ would lie on a plane perpendicular to the page and is not plotted for clarity.

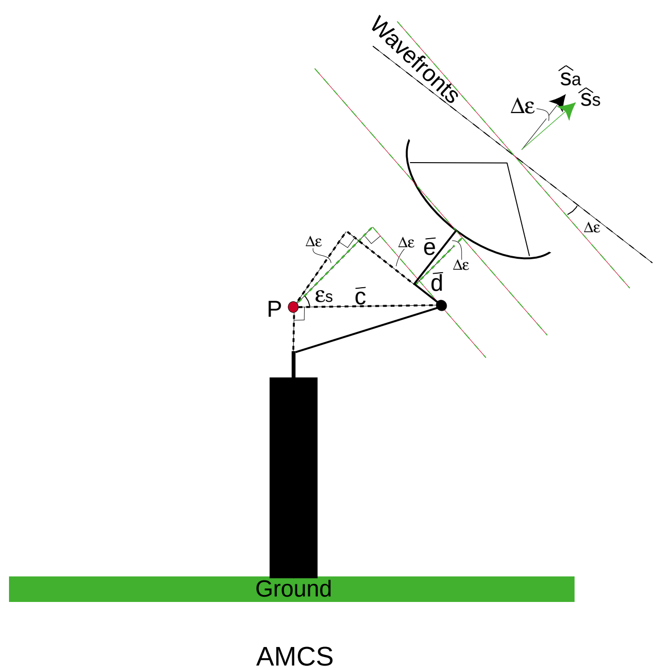

Figure 3 Schematic representation of the geometry of the AMCS baseline model; mispointing

The AMCS phase delay is (Appendix A):

$$\begin{aligned}\hat{s}_s \cdot \bar{a} &= \hat{s}_s \cdot (\bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f}) \\ &= \hat{s}_s \cdot \bar{b} \,+\, |\bar{c}| \cos\varepsilon_s \cos\Delta\phi \\ &\qquad +\, |\bar{d}| \,(\sin\varepsilon_s \cos\varepsilon_a -\, \cos\varepsilon_s \sin\varepsilon_a \cos\Delta\phi) \\ &\qquad +\, (|\bar{e}| + |\bar{f}|)\,(\cos\varepsilon_s \cos\varepsilon_a \cos\Delta\phi +\, \sin\varepsilon_s \sin\varepsilon_a)\end{aligned} \tag{3}$$

Geometric model parameters and errors

Our current best estimates of these model parameters (baseline, azimuth and elevation axis offsets, "antenna arm" and "focal distance") are as follow: (1) baseline vector $\bar{b}$ components (Δe, Δn, Δu; in the sense AMCS phase center minus GPS phase center) are: -20.820, 8.394 and -1.866 m for L1 and -20.820, 8.394 and -1.890 m for L2; (2) azimuth axis offset $|\bar{c}| = 264 \pm 5$ mm; (3) elevation axis offset $|\bar{d}| = 50 \pm 30$ mm; (4) antenna arm $|\bar{e}| = 500$ mm; (5) focal distance $|\bar{f}| = 1$ m. These parameters were measured by Per and are based on estimates from a 24 hr run AMCS residuals (1), ruler measurements (2), and eyeball (3), (4) and (5). The local survey should provide improved values for all these parameters.

The comparatively short length of $\bar{d}$ allows further simplification of (3) by making use of the trigonometric formula $\sin(A - B) = \sin A \cos B - \cos A \sin B$, and the approximation $\cos A = 1$ and $\sin A = A$ valid to first order in A for small angles,

$$|\bar{d}| \,(\sin\varepsilon_s \cos\varepsilon_a -\, \cos\varepsilon_s \sin\varepsilon_a \cos\Delta\phi) \simeq |\bar{d}| \, \sin(\varepsilon_s - \varepsilon_a) = -|\bar{d}| \, \sin\Delta\varepsilon = -|\bar{d}| \,\Delta\varepsilon \tag{4}$$

Note that the azimuth dependence has vanished from this term. The loss of accuracy in expressing the mispointing $|\bar{d}|$-term as in (4) is sub-mm. For example, if we consider the case $\Delta\varepsilon = 0$ (i.e., $\varepsilon_a = \varepsilon_s = \varepsilon$), then $|\bar{d}| \,(\sin\varepsilon_s \cos\varepsilon_a -\, \cos\varepsilon_s \sin\varepsilon_a \cos\Delta\phi) = |\bar{d}| \,(\sin\varepsilon \cos\varepsilon \,(1 - \cos\Delta\phi))$, which for a given $\Delta\phi$ has its maximum value at $\varepsilon = 45°$. At this elevation, the error introduced by not considering this term is below 1 mm for $\Delta\phi < 17°$, which is always true. (The half-power beam width of the AMCS antenna is about 8°.) On ther other hand, if we consider the case $\Delta\phi = 0$, then $|\bar{d}| \,(\sin\varepsilon_s \cos\varepsilon_a -\, \cos\varepsilon_s \sin\varepsilon_a \cos\Delta\phi) = -|\bar{d}| \, \sin\Delta\varepsilon$, which differs from $|\bar{d}| \,\Delta\varepsilon$ by less than 1 mm for $\Delta\varepsilon < 29°$, which is always true. In summary, this simplification is always accurate at the mm-level and the systematic error introduced by this elevation axis offset term amounts to an error of the order of 1 mm per degree of elevation mispointing offset.

The other terms in (3), namely the $|\bar{c}|$-, $|\bar{e}|$- and $|\bar{f}|$-terms, cannot be generally simplified without loss of mm-accuracy. For example, at $\varepsilon_s = 0$, an approximation $\cos\Delta\phi = 1$ in the $|\bar{c}|$-term is below 1 mm only for $\Delta\phi < 5°$. Note that this $|\bar{c}|$-term presents no elevation mispointing dependence. As for the $(|\bar{e}| + |\bar{f}|)$-term, if we consider the case $\varepsilon_a = \varepsilon_s = 0°$, then $(|\bar{e}| + |\bar{f}|)(\cos\varepsilon_s \cos\varepsilon_a \cos\Delta\phi + \sin\varepsilon_s \sin\varepsilon_a) = (|\bar{e}| + |\bar{f}|)\cos\Delta\phi$. At this elevation, an approximation $\cos\Delta\phi = 1$ is below 1 mm only for $\Delta\phi < 2°$. On the other hand, if we consider the case $\Delta\phi = 0$, and making use of the trigonometric formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$, $(|\bar{e}| + |\bar{f}|)(\cos\varepsilon_s \cos\varepsilon_a \cos\Delta\phi + \sin\varepsilon_s \sin\varepsilon_a) = (|\bar{e}| + |\bar{f}|)\cos\Delta\varepsilon$, for which an approximation $\cos\Delta\varepsilon = 1$ is below 1 mm only for $\Delta\varepsilon < 2°$.

Summary of the AMCS geometric model

From (2)-(4), the perfect pointing phase delay is,

$$\hat{s}_s \cdot \bar{a} = \hat{s}_s \cdot \bar{b} + |\bar{c}| \cos\varepsilon_s + 0 + |\bar{e}| + |\bar{f}| \tag{5}$$

and the mispointing phase delay is,

$$\begin{aligned} \hat{s}_s \cdot \bar{a} = \hat{s}_s \cdot \bar{b} \; + \; & |\bar{c}| \cos\varepsilon_s \cos\Delta\phi - |\bar{d}|\,\Delta\varepsilon \\ & + (|\bar{e}| + |\bar{f}|)(\cos\varepsilon_s \cos\varepsilon_a \cos\Delta\phi + \sin\varepsilon_s \sin\varepsilon_a) \end{aligned} \tag{6}$$

By comparing (5) and (6), it can be seen that as a result of mispointing, an error term arises that is proportional to:

$$|\bar{c}| \cos\varepsilon_s (1 - \cos\Delta\phi) + |\bar{d}|\Delta\varepsilon + (|\bar{e}| + |\bar{f}|)(1 - \cos\varepsilon_s \cos\varepsilon_a \cos\Delta\phi - \sin\varepsilon_s \sin\varepsilon_a) \tag{7}$$

These mispointing terms have to be included in the AMCS phase model. The systematic error introduced by the azimuth axis offset (term $|\bar{c}|$ in (7)) varies sinusoidally with cosine of the satellite elevation angle from a value of 0 mm at $\varepsilon_s = 90°$ to a maximum value at $\varepsilon_s = 0°$. At the latter elevation, this term reaches 1 mm level at an azimuth mispointing offset $\Delta\phi = 5°$, and increases thereof (Figure 4). This term presents no elevation mispointing dependence. The systematic error introduced by the elevation axis offset (term $|\bar{d}|$ in (7)) amounts to an error of the order of 1 mm per degree of elevation mispointing offset (Figure 4). This term presents no azimuthal mispointing dependence. The systematic error introduced by the $(|\bar{e}| + |\bar{f}|)$ term in (7), reaches 1 mm level at an elevation mispointing offset $\Delta\varepsilon = 2°$ when $\Delta\phi = 0°$ and, when $\Delta\varepsilon = 0°$, varies sinusoidally with the squared of the cosine elevation angle from a value of 0 mm at $\varepsilon_s = 90°$ to a maximum value at $\varepsilon_s = 0°$. At the latter elevation, this term reaches 1 mm level at an azimuth mispointing offset $\Delta\phi = 2°$, and increases thereof (Figure 4). For small ($\Delta\varepsilon < 1°$) AMCS mispointing angles, (7) reduces to $|\bar{d}|\Delta\varepsilon$.

By construction, this antenna geometric model assumes that the azimuth axis is vertical and the elevation axis is in a plane perperdicular to the azimuth axis. The

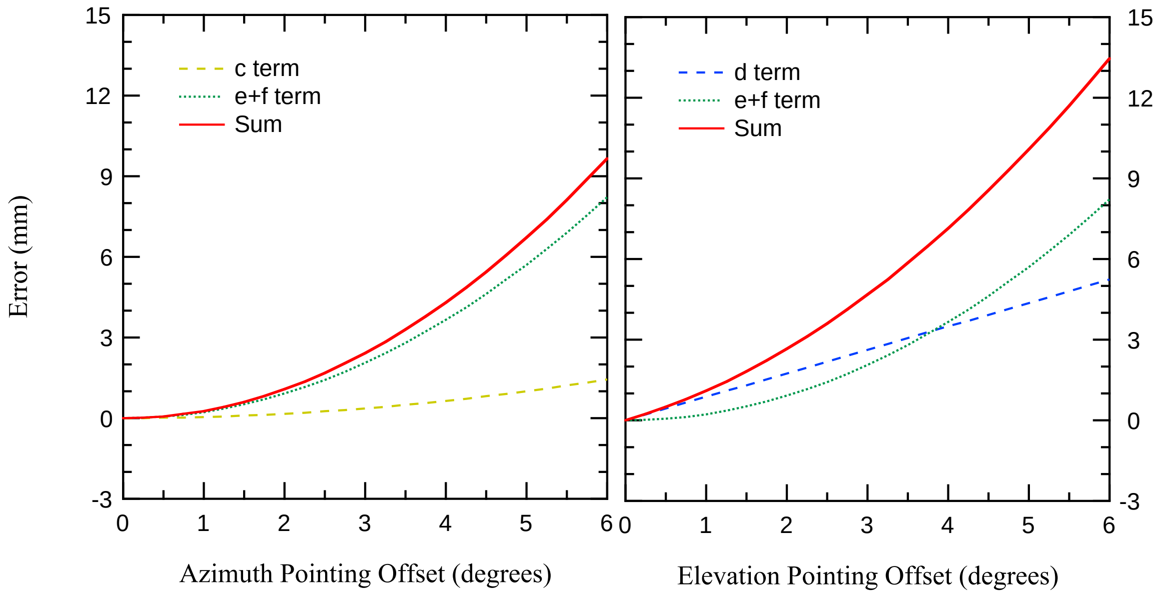

Figure 4 Mispointing error after (7) for (left) $\Delta\varepsilon$ and $\varepsilon_s = 0°$, and (right) $\Delta\phi = 0°$.

model does not account for effects suchs as (elastic) deformation and/or wearing of the antenna elements, nor includes phase variations due to the AMCS antenna phase pattern. Two additional, related effects that need to be considered in modeling the AMCS phase delays are "feed rotation" and atmospheric refraction.

Appendix A: AMCS mispointing phase delay

To derive (3), we express vectors $\hat{s}_s, \bar{c}, \bar{d}, \bar{e}$ and $\bar{f}$ in a local coordinate system. For simplicity, we adopt a coordinate system where the z-axis is vertical, the x-axis horizontal and the satellite lies in the x-z plane (e.g., Figures 2 and 3). The components of the satellite unit vector in this coordinate system are

$$\hat{s}_s = (\cos\varepsilon_s, 0, \sin\varepsilon_s) \qquad (A-1)$$

As for the antenna vectors, first we consider a mispointing angle in elevation and then azimuth. The components of the antenna vectors in this local coordinate system are

$$\begin{aligned} \bar{c}' &= |\bar{c}|\,(1,0,0) \\ \bar{d}' &= |\bar{d}|\,(-\sin\varepsilon_a, 0, \cos\varepsilon_a) \\ \bar{e}' + \bar{f}' &= |\bar{e}+\bar{f}|\,(\cos\varepsilon_a, 0, \sin\varepsilon_a) \end{aligned} \qquad (A-2)$$

where $'$ refers to only elevation mispointing. To also account for an azimuth mispointing angle $\Delta\phi = \phi_a - \phi_s$, we give these primed antenna vectors a single right-handed rotation of angle $\Delta\phi$ about the z-axis (a positive rotation being counterclockwise when looking toward the origin from the positive axis) using

$$R_z(\Delta\phi) = \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi & 0 \\ -\sin\Delta\phi & \cos\Delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (A-3)$$

to obtain

$$\begin{aligned} \bar{c} &= R_z\bar{c}' = |\bar{c}|\,(\cos\Delta\phi, -\sin\Delta\phi, 0) \\ \bar{d} &= R_z\bar{d}' = |\bar{d}|\,(-\sin\varepsilon_a\cos\Delta\phi, \sin\varepsilon_a\sin\Delta\phi, \cos\varepsilon_a) \\ \bar{e}+\bar{f} &= R_z(\bar{e}'+\bar{f}') = |\bar{e}+\bar{f}|\,(\cos\varepsilon_a\cos\Delta\phi, -\cos\varepsilon_a\sin\Delta\phi, \sin\varepsilon_a) \end{aligned} \qquad (A-4)$$

and the dot product of $\hat{s}_s$ and these vectors are

$$\begin{aligned} \hat{s}_s\cdot\bar{c} &= |\bar{c}|\cos\varepsilon_s\cos\Delta\phi \\ \hat{s}_s\cdot\bar{d} &= |\bar{d}|\,(\sin\varepsilon_s\cos\varepsilon_a - \cos\varepsilon_s\sin\varepsilon_a\cos\Delta\phi) \\ \hat{s}_s\cdot(\bar{e}+\bar{f}) &= (|\bar{e}|+|\bar{f}|)\,(\cos\varepsilon_s\cos\varepsilon_a\cos\Delta\phi + \sin\varepsilon_s\sin\varepsilon_a) \end{aligned} \qquad (A-5)$$

as used in (3).